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NUMERICAL STUDY OF CHARGED PARTICLE HEATING BY PLASMA TURBULENCE†

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ABSTRACT

We report numerical calculations of the heating of charged particles by turbulent electric fields. The fields are given, and are not required to be self-consistently supported by the particles. Three types of wave spectra are treated, each of which is analytically unmanageable: (1) many waves, constant phases and velocities; (2) one wave, randomly interrupted phase; (3) one wave, slowly varying wave number at constant frequency. Large particle energization is found in the last two cases. In all three cases, the results can be qualitatively understood by elementary arguments related to particle "trapping."

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I. INTRODUCTION

Recent experiments on turbulent heating devices[1] have directed attention to details of the processes by which rapidly fluctuating electric fields transfer energy to the thermal motions of a plasma.

The plasma in these devices is usually far from the state in which theorists like to idealize it (uniform, quiescent, noise-free, weakly perturbed, etc.). It is not obvious that analytical attacks on such a plasma are of any use. The situation is at least as complicated as in aerodynamic turbulence theory, where agreed-upon analytical results are rare. So far, the linearized Vlasov industry has shed little light on turbulent heating of a plasma.

In an effort to understand the turbulent heating process we have done computations with an IBM 360/65 on charged particle heating by a superposition of electrostatic waves. The wave spectrum is assumed known, one of the input parameters of the problem along with the initial phase space locations of the particles. It has seemed advisable at this stage artificially to separate the orbit theory from the self-consistent aspects of the problem, and discard the latter.

Our objective has been to achieve a physical understanding of the way given spectra heat (or fail to heat) an assemblage of charged particles. Once it is determined what sorts of spectra heat most efficiently, the next step would appear to be to attempt to tailor these spectra in laboratory devices. Recently developed techniques of measuring field correlations in the Ghz range[2] have improved the chances for being able to do this. It appears to us hopeless to attempt a comprehensive theory of any turbulent heating--analytical or numerical.

II. FORMULATION OF THE PROBLEM

We now remark on the formulation of the heating problem for turbulent electric fields. We shall work throughout with idealized one-dimensional "particles" and fields.

The equation of motion of a particle is

$$\ddot{x}(t) = \dot{v}(t) = \varepsilon(t) \qquad (1)$$

where $\varepsilon(t) \equiv (q/m)E(x(t),t)$. The position and velocity are $x(t)$, $v(t)$, the charge-to-mass ratio is q/m, and $E(x,t)$ is the electric field as a function of x and t. Equation (1) is not generally soluble unless $E(x,t)$ becomes time-independent in some coordinate frame.

The trajectory of one particle is often of little interest, and our attention may be directed to one of two kinds of ensemble averages:

(1) Averages over an ensemble of initial coordinates $x(0)$, $v(0)$ for one given $E(x,t)$.

(2) Averages over an ensemble of electric fields for one given initial $x(0)$, $v(0)$. For convenience, we may choose $x(0) = 0$.

The two averages are not the same, and it has seemed to us generally preferable to compute the second.[3]

The electric field is generally expressed as

$$E(x,t) = \sum_i E_i \cos(k_i x - \omega_i t + \phi_i), \qquad (2)$$

with given field amplitudes E_i, wave numbers k_i and phases ϕ_i. The ensemble averaging is over the phases ϕ_i, and is indicated by a bracket < >. The E_i, k_i, ω_i, $x(0)$ are not averaged over, as a rule. The average is therefore over what happens to identical particles released in plasmas with identical spectra, but with phases which differ from plasma to plasma.

We are interested in two quantities more than the others, the average velocity $\langle v \rangle$, and the average spread in velocities which we call the kinetic "temperature" $\langle (v-\langle v\rangle)^2 \rangle$. The latter always starts out at zero, and one way of formulating the goals of the turbulent heating problem is to make $\langle (v-\langle v\rangle)^2 \rangle$ as large as possible as fast as possible with a fixed amount of field energy available.

III. RESULTS

We usually work in dimensionless[4] variables for which the equations of motion take the form

$$\dot{x}(t) = v(t)$$

$$\dot{v}(t) = -\sum_{i=1}^{N} \varepsilon_i \cos(k_i x(t) - \omega_i t + \phi_i). \qquad (3)$$

We turn now to a summary of some of the results of the numerical solution to Eqs. (3), first making some general remarks which are useful in interpreting the results.

A. ONE WAVE, CONSTANT PHASE

A useful concept has proved to be the "trapping width" of a wave. The system (3) is only analytically solvable for the case N=1, as is well known. The solution is expressible in terms of Jacobian elliptic functions, and the properties of the solution can be summarized as follows. For $v(0)$ far from the phase velocity ω_1/k_1, the orbit $x(t)$, $v(t)$ is only slightly deflected from its free flight value $x(0) + tv(0), v(0)$. Those particles whose orbits are greatly modified are those whose velocities lie within a distance $\pm 2\sqrt{\varepsilon_1/k_1}$ of ω_1/k_1. We call this range of velocity space the "trapping range" of the wave. In the frame which moves with velocity ω_1/k_1 (the "wave frame"), the particle's energy is $w \equiv (v(0))^2/2 - (\varepsilon_1/k_1)\cos\phi_1$. If $w < \varepsilon_1/k_1$, the particle is "trapped," i.e., its orbit has turning points in velocity in the wave frame. Trapped particles oscillate in velocity in the wave frame with a period which ranges from the "trapping time" $2\pi/\sqrt{k_1\varepsilon_1}$ to infinity when w goes from its minimum value to ε_1/k_1. The maximum velocity increment a particle can acquire from a single wave is of the order of $4\sqrt{\varepsilon_1/k_1}$.

B. SEVERAL WAVES, CONSTANT PHASES

This simple interpretation of particle trapping fails when N>1, but it is still useful to think of the waves as occupying a "trapping range," $\frac{\omega_i}{k_i} - 2\sqrt{\frac{\varepsilon_i}{k_i}} \lesssim v \lesssim \frac{\omega_i}{k_i} + 2\sqrt{\frac{\varepsilon_i}{k_i}}$ in the velocity space. By repeated numerical solution of Eqs. (3), we have established that particles will generally make long hops in velocity space if, and only if, the two regions of velocity space are connected by the overlapping trapping widths of waves.

One immediate consequence of this is that many little waves will heat the particles hotter than one big one, given a fixed amount of total wave energy. This is because, as the number of waves increases, the amplitude falls off as $N^{-1/2}$, so the trapping width falls off as $N^{-1/4}$. The total width of velocity that can be spanned by the waves goes as $N\cdot N^{-1/4} = N^{3/4}$.

This effect is illustrated in Fig. 1, where $\langle (v-\langle v\rangle)^2 \rangle$ is plotted against time for one wave and ten waves, with the same total value of $\sum_{i=1}^{N} \varepsilon_i^2$. The particles are heated about a factor of three hotter in the ten wave case.

Discussion of the Constant Phase Cases

Beyond the simple conclusion that particles will only go where there are waves in velocity space, there have been few surprises.

We have not had noteworthy success in fitting the results of the computations with such analytical theory as there is for this problem.[5] For example, the Fokker-Planck equation

$$\frac{\partial f(v,t)}{\partial t} = \frac{\partial}{\partial v}\left[D(v,t) \frac{\partial}{\partial v} f(v,t)\right], \qquad (4)$$

where $f(v,t)$ is the velocity distribution function and $D(v,t)$ is the diffusion coefficient

$$D(v,t) \equiv \pi \sum_i \varepsilon_i^2 \, \delta(k_i v - \omega_i) \geq 0, \qquad (5)$$

has few analytical consequences which can be extracted, and such consequences as it does have apply to the case where k is a continuously-distributed, rather than a discrete, variable (such as the development of a "plateau" in the distribution f; it is worth remarking that Eqs. (4) and (5) do not predict the development of a plateau). The methods used to derive (4) and (5) are intrinsically incapable of treating situations in which the wave velocity spacing is comparable to, or greater than, the trapping widths of the waves.

An analytical framework capable of acting as a good zeroth approximation to this situation would be very welcome. We have experimented some with a master equation approach, where

$$\frac{dn_i(t)}{dt} = \sum_j \left\{ A_{ji}\, n_j(t) - A_{ij}\, n_i(t) \right\}. \qquad (6)$$

Here, $n_i(t)$ is the number of particles within the trapping range of the i^{th} wave, and A_{ij} is a transition probability per unit time for the particle's jumping from wave i to wave j. This approach still looks hopeful to us, but we have no quantitative success to report.

C. ONE WAVE, RANDOMLY VARYING PHASE

In diagnosing turbulent heating experiments, one frequently obtains probe signals which represent coherent oscillations for several cycles, then are interrupted in an apparently random way, and continue to reappear and disappear. We have attempted to simulate this type of oscillation by setting

$$\ddot{x}(t) = \dot{v}(t) = -\varepsilon \cos\,[k_o\, x(t) - \omega_o t + \phi(t)], \qquad (7)$$

where ε, k_o, and ω_o are again constants, but $\phi(t)$ is a random number modulo 2π which jumps to a new value whenever $t = n\tau$, $n = 1, 2, 3, \ldots$, with a fixed interval of constancy τ. The heating which results from Eq. (7) turns out to be more efficient, as well as theoretically more interesting, than the heating in the constant phase cases.

In Fig. 2, we see the temperature vs. time for different intervals of τ. The ensemble average is over a hundred initial values of the phases. The trapping width $4\sqrt{\varepsilon/k_o}$ is about 4 for all cases. The heating to be expected for $\phi(t)$ = a constant for all t would give a temperature of about unity. The temperature reached for finite τ is clearly greater than that. (The constant-phase limit is $\tau = \infty$.)

Can we make a qualitative connection between the heating expected here and the well-understood results of the previous (constant-phase) cases? The answer is yes, if we characterize the electric field by its auto-correlation function,[6]

$$\langle C(\Delta x,\Delta t)\rangle \equiv$$

$$\varepsilon^2 \lim_{\substack{L\to\infty \\ T\to\infty}} \int_{-L/2}^{L/2} \frac{dx}{L} \int_{-T/2}^{T/2} \frac{dt}{T} \Big\langle\!\Big\langle \cos\,[k_o x - \omega_o t + \phi(t)]$$

$$\cdot \cos\,[k_o(x+\Delta x) - \omega_o(t+\Delta t) + \phi(t+\Delta t)] \Big\rangle\!\Big\rangle$$

$$= \int dk \int d\omega\, S(k,\omega) \exp\,[i(k\Delta x - \omega\Delta t)]. \qquad (8)$$

(Double brackets mean an ensemble average over initial phases.)

The spectral density $S(k,\omega)$ is proportional to the average energy density per unit wave number per unit frequency in the electrostatic wave field. We can think of those points in k,ω space where $S(k,\omega)$ is large as being the points where the waves are, on the average.

For constant phase ($\tau = \infty$), we have

$$S(k,\omega) = S_\infty(k,\omega)$$

$$= \frac{\varepsilon^2}{4}\,[\delta(k-k_o)\,\delta(\omega-\omega_o) + \delta(k+k_o)\,\delta(\omega+\omega_o)]. \qquad (9a)$$

Thus the only waves are at k_o, ω_o, and $-k_o$, $-\omega_o$.

For finite τ, on the other hand,

$$S(k,\omega) = \frac{\varepsilon^2}{2\pi\tau} \left\{ \frac{\delta(k-k_o)\sin^2((\omega-\omega_o)\tau/2)}{(\omega-\omega_o)^2} + \frac{\delta(k+k_o)\sin^2((\omega+\omega_o)\tau/2)}{(\omega+\omega_o)^2} \right\}. \qquad (9b)$$

Though for $\tau \to \infty$, $S(k,\omega) \longrightarrow S_\infty(k,\omega)$ uniformly, we see that for finite τ, the effect is that of adding a distribution of waves whose wavelength is still $2\pi/k_o$, but which are continuously distributed in frequency about $\omega \equiv \pm \omega_0$, over a region of ω-space whose width can be estimated as $\Delta\omega \sim 2\pi/\tau$. This width corresponds to a spread in phase velocities of order $\Delta\omega/k_0 \sim 2\pi/k_0\tau$, which ranges from about 0.5 for $\tau = 64$ to 30 for $\tau = 1$.

It is reasonable to expect the particle heating to increase in magnitude as the width of the effective phase velocity spectrum increases (τ decreases), and this is apparently what happens in Fig. 2. A countervailing effect is that as τ decreases, the maximum value of the additional term in Eq. (9b) decreases, thus weakening the overall field strength of the additional spectrum. This means a slower migration of particles over the region of velocity space occupied by ω/k, or a longer time required to attain the maximum heating. This is apparently what has occurred for $\tau = 1$ in Fig. 2, where the computer program ceases to be accurate at a time when the temperature is far from its maximum value.

In Fig. 3, we show the velocity space distribution function at $t = 300$ for each of the values of τ. The distribution of $v - \langle v \rangle$ becomes approximately symmetrical, as it does for the constant-phase cases.

D. ONE WAVE, DECREASING WAVE NUMBER[7,8]

An important practical situation is that in which an electrostatic wave propagates into a region of increasing density at constant frequency, with an attendant decrease of wave number. This situation can be simulated by the equation

$$\ddot{x} \equiv \dot{v} = \varepsilon \cos\left[k(x)x - \omega t + \phi\right], \qquad (10)$$

where $k(x)$ is a slowly decreasing function of x. For definiteness, we chose $k(x) = (1+\alpha x)^{-1} k_0$, where α is a positive constant much less than one.

Swift[7] has suggested that since the local phase velocity of the electric field in Eq. (10) increases slowly to the right, an initially trapped right-travelling particle may be able to ride the wave up in velocity, thereby being accelerated to a much higher energy for given ε than could be attained by k = const. We now show how numerical calculations show this conjecture to be correct.

Figure 4 shows the velocity as a function of time seen by a single particle, to illustrate this effect. The solid line is the velocity of the particle; the broken line is $\omega(k + x\,dk/dx)^{-1}$, which is the instantaneous phase velocity seen by the particle at position x. Not much happens to the particle before it becomes "trapped" between $t = 100$ and 200, and not much happens to it after it becomes "untrapped" around $t = 800$, but while it is trapped, it acquires a factor of 16 in energy from a field strength ε that would produce a net energy gain of at most ~ 1.5 if k were constant.

Figure 5 shows the velocity distribution as histograms for various values of α. The limit $\alpha \equiv 0$ is the case of $k \equiv$ const. It will be clear that not all the particles will ride the wave up in energy, and those that become "untrapped" experience no significant acceleration thereafter. The smaller α is, the longer the particle can remain trapped, because small α implies a very gentle acceleration of the phase velocity, and the trapped particles' behavior is essentially adiabatic. This accounts for the more efficient acceleration in the small α cases.

This method of producing a few very energetic particles with small field strengths is believed by us to have geophysical implications.[8]

IV. SUGGESTIONS FOR THE FUTURE

Two directions appear to be fruitful for generalizing these numerical techniques. They are closely related.

(1) Generalization to more than one dimension (two dimensions are probably more feasible than three).

(2) Inclusion of d.c. magnetic fields in the equations of motion.

One important effect in the two and three dimensional cases is connected with the fact that the condition for strong interaction of a wave and a particle is much less restrictive.[9] It is that

$$\omega \cong \vec{k} \cdot \vec{v},$$

rather than $\omega \cong kv$. This enables particles to be accelerated to much greater velocities than the maximum value of $\omega/|\vec{k}|$. For the one dimensional case, the maximum value of ω/k always sets the upper limit on speeds to which particles can be accelerated.

Secondly, the addition of the possibility of cyclotron resonance[10] to the trapping acceleration described here will greatly enhance the possibilities for particle orbits to differ greatly from their conventional perturbation-theroretic values.

It does seem important at this stage to try to keep the problems as simple and easily understandable as possible, and to try to keep them as tied to concrete experiments as possible. For this, more electric-field diagnostics (spectral densities, autocorrelations, etc.) are needed than are now available on most of the turbulence experiments. It would be easy for theory and experiment to separate at this point just as they have for other types of plasma machines, with the same resulting obscurity and confusion.

V. ACKNOWLEDGEMENTS

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FOOTNOTES

1. See, for example,
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3. The reason for this is that it is difficult to be sure that any one particular E(x,t) is representative or "typical" of various turbulent field configurations.
4. A wide variety of dimensionless variables is possible, corresponding to the various kinds of oscillations which have been observed. One way to interpret Eqs. (3), for example, is to imagine a background plasma which supports electron plasma oscillations. Lengths are measured in units of the Debye length for that plasma, and times in inverse plasma frequencies; velocities are measured in the ratio of these. The test particles which obey Eqs. (3) are electrons, and the $\sum_{i=1}^{N} \varepsilon_i^2$ is the ratio of electric field energy to the thermal energy density of the background plasma, and thus measures the "noise". For definiteness, the ω_i and k_i are chosen to obey the (dimensionless) Bohm-Gross dispersion relation,
   $$\omega_i^2 \simeq 1 + 3k_i^2 + 6k_i^4 + \dots \quad .$$
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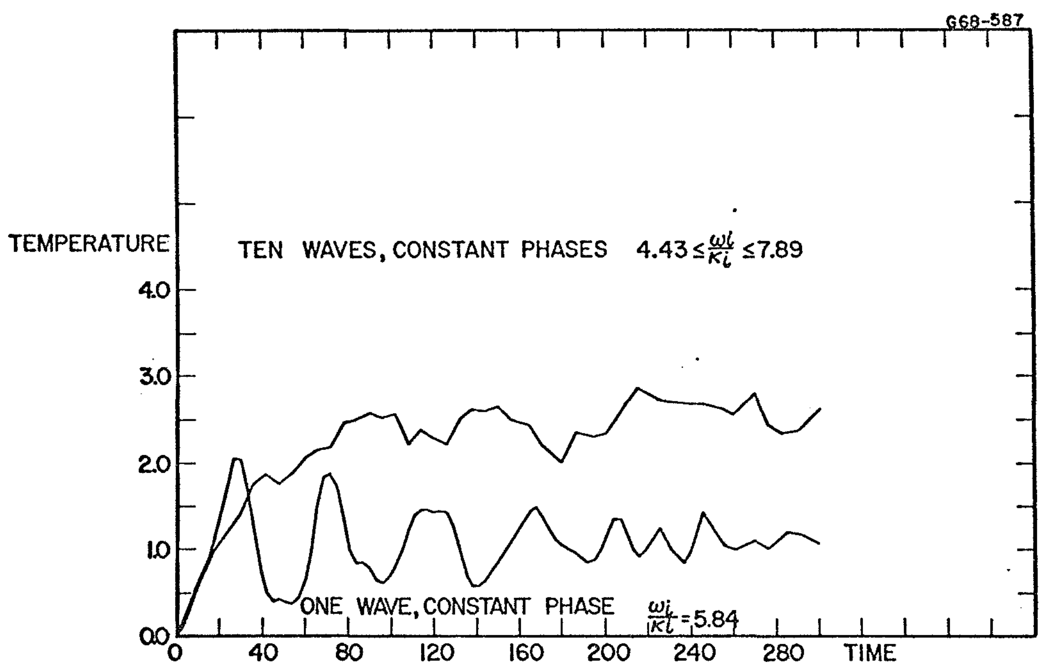

Figure 1 Temperature as a function of time for N = 1 and N = 10. In both cases, v(0) = 4.5 initially for all particles and $\sum_{i=1}^{N} \varepsilon_i^2 = 0.025$. In the one-wave case, k = 0.18, ω = 1.052, and ε = 0.1581. For the ten-wave case, k ranges from 0.13 to 0.25, ω ranges from 1.026 to 1.110, and ε = 0.05 for all ten waves. Ensemble average is over 100 "particles".

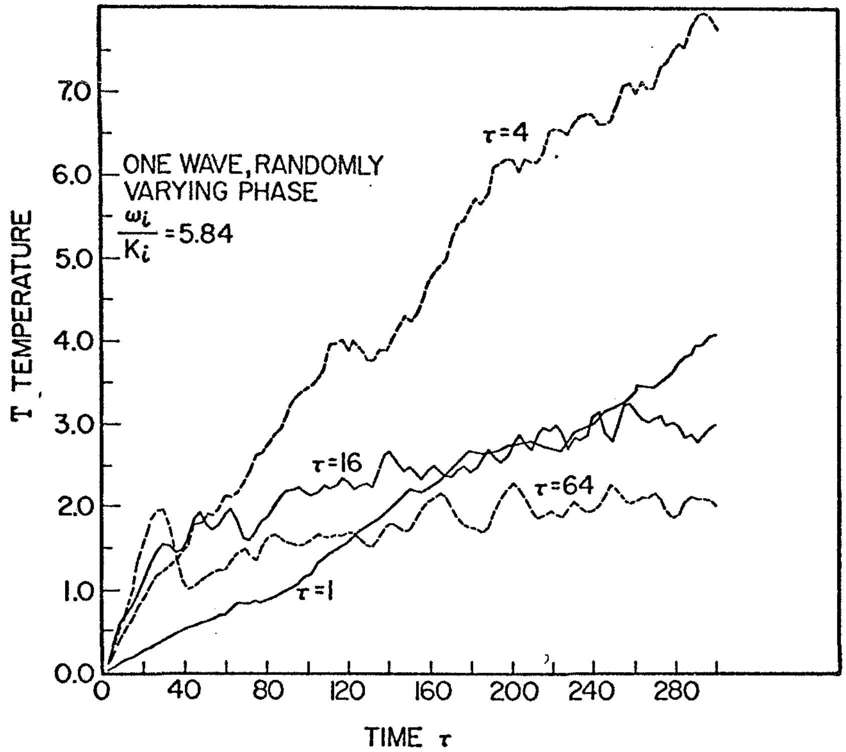

Figure 2 Temperature as a function of time for N = 1 and the phase $\phi(t)$ a random variable. In all cases, k = 0.18, ω = 1.052, and ε = 0.1581. The only quantity being varied is τ, the interval over which $\phi(t)$ remains constant before it is interrupted. Ensemble average is over 100 "particles".

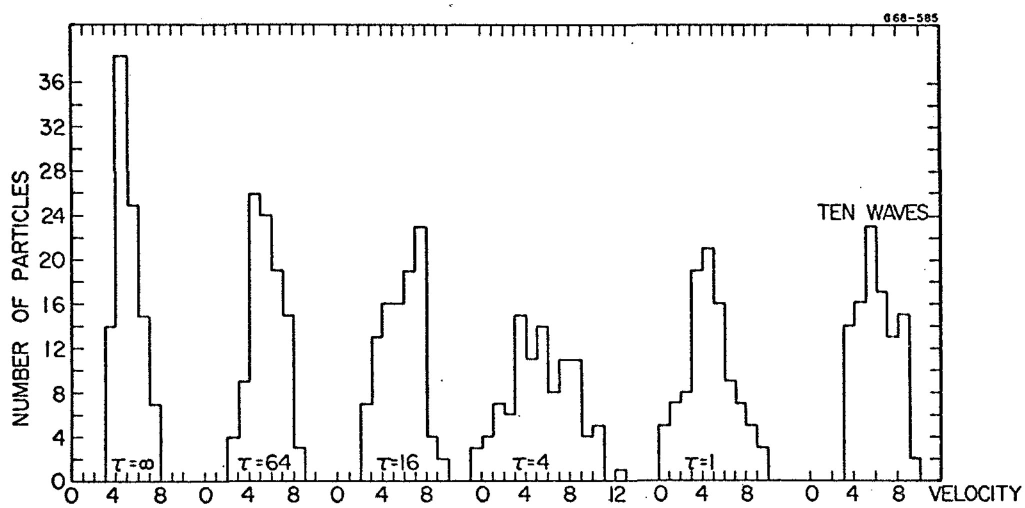

Figure 3 The number of "particles" per unit velocity range at time t = 300 vs. velocity. "$\tau = \infty$" and "ten waves" graphs correspond to the N = 1 and N = 10 cases, respectively, of Figure 1. The finite τ graphs correspond to the respective cases in Figure 2. The arrows along the velocity axes indicate the boundaries of the trapping ranges in the constant phase cases.

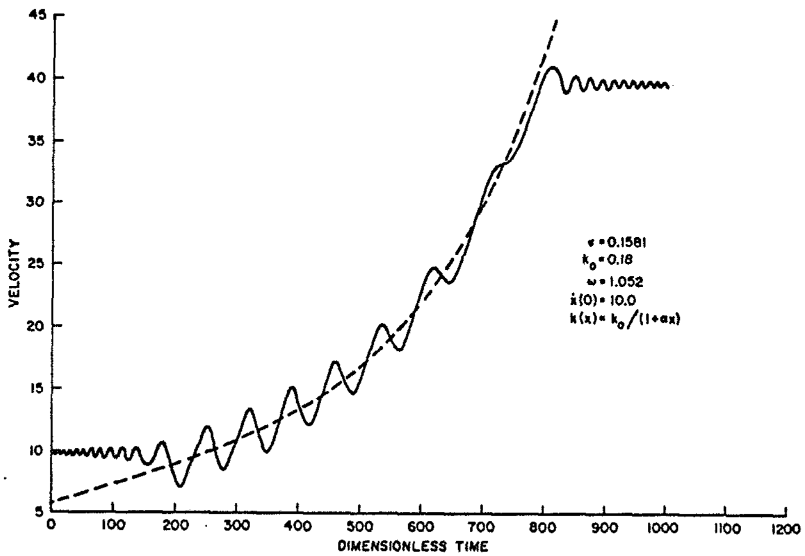

Figure 4 Velocity as a function of time for a typical particle, with $\dot{x}(0) = 10.0$ and $\frac{\omega}{k} = \frac{1.052}{0.18}\ (1 + 0.000125\ x)$. Note that the particle experiences little acceleration after it "falls off" the wave at about $\dot{x}(t) = 40$. The broken line is the effective "phase velocity" evaluated at the instantaneous position of the particle. (Time step here was chosen to be $\Delta t = 0.25$ and 0.50. The results agreed to better than three significant figures up to $t = 1000$.)

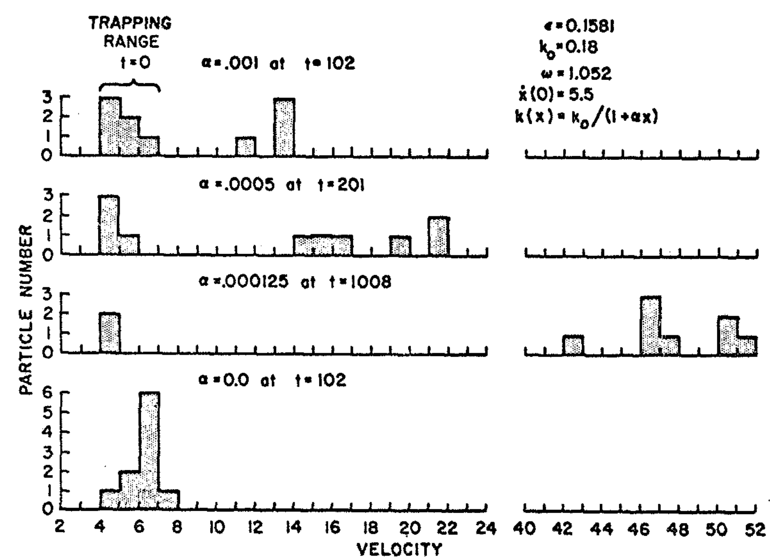

Figure 5 Number of particles for unit velocity range at large times for $\dot{x}(0) = 5.5$, and $\frac{\omega}{k} = \frac{1.052}{0.18}\ (1 + \alpha x)$ for $\alpha = 0,\ 1.25 \times 10^{-4}$, 5×10^{-4}, and 10^{-3}. The smaller α, the longer the particle can remain trapped, if the wave train is infinitely long and completely coherent. However, for α strictly zero, little acceleration occurs. After the times indicated, the particles have all become untrapped in the $\alpha > 0$ cases.